

An algebro-geometric solution for a Hamiltonian system with application to dispersive long wave equation

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By using an iterative algebraic method, we derive from a spectral problem a hierarchy of nonlinear evolution equations associated with dispersive long wave equation. It is shown that the hierarchy is integrable in Liouville sense and possesses bi-Hamiltonian structure. Two commutators, with zero curvature and Lax representations, for the hierarchy are constructed, respectively, by using two different systematic methods. Under a Bargmann constraint the spectral is nonlinearized to a completely integrable finite dimensional Hamiltonian system. By introducing the Abel–Jacobi coordinates, an algebro-geometric solution for the dispersive long wave equation is derived by resorting to the Riemann theta function. © 2005 American Institute of Physics. [DOI: 10.1063/1.1857064]

I. INTRODUCTION

The development of a integrable system representing well-known physical phenomena has shown to be meaningful and valuable. For instance, the illustration of the bi-Hamiltonian structure of a system of partial differential equations is proven to be a direct and elegant method in establishing the complete integrability of the system.^{1–5} If a set of partial differential equations can be formulated as a Hamiltonian system in two distinct but compatible ways, Magri had proven in Ref. 1 that this gives rise to an infinite sequence of conserved Hamiltonians which are in involution with respect to one of those two symplectic structures. Recently, two constructive approaches that can handle both finite-dimensional and infinite-dimensional integrable Hamiltonian systems were successfully developed. The first approach is based on the trace identity,^{6,7} which is effective in constructing the infinite-dimensional Liouville integrable Hamiltonian systems. Starting from a properly defined isospectral problem, many integrable hierarchies and their Hamiltonian structures (e.g., AKNS, TC, TA, BPT, Yang) had been obtained by applying this method.^{6–10} The second one used nonlinearization technique,^{11,12} which has also been proven to be powerful for obtaining new finite-dimensional integrable Hamiltonian systems from various soliton hierarchies. Under the Bargmann or Neumann constraints on the potentials and the eigenvalues, which play a central role in the process of nonlinearization, the related eigenvalue problem can be nonlinearized as a finite-dimensional completely integrable system. This covers the eigenvalue problems associated with the well-known soliton hierarchies such as the KdV, AKNS, Jaulent–Miodek, Kaup–Newell, etc.^{10–17} An advantage of this method is that its solution to the soliton equation associated with an eigenvalue problem can be reduced to solving a compatible system of nonlinear ordinary differ-

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ential equations.^{9–15} This approach has now been further developed to a general method for handling higher order constraints associated with infinitely many hierarchies of finite-dimensional integrable Hamiltonian systems.^{18–20}

The algebro-geometric method is an analogue of inverse scattering transformation. It was first developed by Dubrovin, Its, Matveev, and Belokolos *et al.*^{21–23} and more recently by Gesztesy and Holden.²⁴ The method can derive an important class of exact solutions, which is called algebro-geometric solution, to the soliton equation such as KdV equation, sine-Gordon equation, and Schrödinger equation. In the degenerated case the method gives the soliton solution and elliptic function solution. Recently, an alternate approach based on the nonlinearization technique of Lax pairs or the restricted flow technique has been proposed.^{25–27} By this method or other methods,^{28–33} algebro-geometric solutions for (1+1)- and (2+1)-dimensional soliton equations can now be obtained.^{23–33}

The dispersive long wave (DLW) equation

$$\begin{aligned}u_t &= 2uu_x + 2v_x + u_{xx}, \\v_t &= 2(uv)_x - v_{xx}\end{aligned}\tag{1.1}$$

was first derived by Whitham and Broer for simulating dispersive waves in shallow water.^{34,35} Its symmetries, conservation laws, similarity reductions, painlevé property and soliton solutions had been fully discussed.^{36–38} In recent years, the following spectral problem associated with the DLW equation (1.1),

$$\psi_x = U\psi = \begin{pmatrix} -\frac{1}{2}(\lambda - u) & -v \\ 1 & \frac{1}{2}(\lambda - u) \end{pmatrix} \psi,\tag{1.2}$$

has been proposed.^{13,39} Under two different constraints between the potentials and eigenfunctions, the nonlinearization of the spectral problem gives two kinds of finite-dimensional completely integrable systems.¹³ By using gauge transformation

$$\tilde{\phi}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi, \quad \lambda = 2i\eta, \quad u = 2\alpha r, \quad v = q + \beta r_x,$$

the spectral problem (1.1) can be shown to be in close analogy to the classical Boussinesq spectral problem,

$$\tilde{\psi}_x = \begin{pmatrix} -i\eta - \alpha r & q + \beta r_x \\ -1 & i\eta + \alpha r \end{pmatrix} \tilde{\psi}.$$

Therefore, the DLW hierarchy is intimately related to the Boussinesq hierarchy, whose algebro-geometric solutions can be found from the work of Gesztesy and Holden.²⁴

In this paper, we would investigate some aspects related to the DLW equation (1.1) and its spectral problem (1.2). In Sec. II starting from a spectral problem, we first derive a hierarchy associated with the DLW equation by using an iterative algebraic method. It will be shown that the hierarchy is integrable in the Liouville sense and its bi-Hamiltonian structure can be established from trace identity. In Sec. III we present a systematic method to construct a zero curvature representation for the hierarchy. In Sec. IV, from the nonlinearization of the spectral problem (1.2), we devise a finite dimensional completely integrable Hamiltonian system under a Bargmann constraint. The Abel-Jacobi coordinates are introduced in Sec. V from which the algebro-geometric solutions for the DLW equations will be derived by resorting to the Riemann theta functions.

II. DLW HIERARCHY AND ITS BI-HAMILTONIAN STRUCTURE

We first recall briefly some notations from the theory of generalized Hamiltonian equations:⁶ Let G be a finite-dimensional Lie algebra over $\mathbb{C}$ and $\tilde{G}$ be the corresponding iterated algebra,

$$\tilde{G} = G \otimes \mathbb{C}[\lambda, \lambda^{-1}],$$

where $\mathbb{C}[\lambda, \lambda^{-1}]$ is the set of Laurent polynomials in λ . Suppose that $\{e_1, \dots, e_p\}$ is a basis of G . The set $\{e_i(n), \dots, e_p(n)\}$ is then a basis of $\tilde{G}$, where $e_i(n) = e_i(0)\lambda^n = e_i\lambda^n$. For $e(n) \in \tilde{G}$, $i = 1, \dots, p$, its gradation is defined by

$$\deg(e_i(n)) = \deg(e_i\lambda^n) = n.$$

Consider in the following a general spectral problem:

$$\psi_x = U(u, \lambda)\psi, \quad (2.1)$$

where $u = (u_1, \dots, u_p)^T \in S$, S is a Schwarz space, and λ is a spectral parameter. The function $U(u, \lambda)$ can be unified to the following representation:

$$U = R + u_1 e_1 + \dots + u_p e_p,$$

where $R, e_1, \dots, e_p \in \tilde{G}$. We define the ranks for ∂, u, λ , and $x \in \tilde{G}$ in such a way that if ab is well defined for any two entities a and b , then

$$\text{rank}(ab) = \text{rank}(a) + \text{rank}(b).$$

We also define the rank of R in such a way that above element U has homogeneous rank, i.e.,

$$\text{rank}(R) = \text{rank}(u_1 e_1) = \dots = \text{rank}(u_p e_p).$$

Finally, we define

$$\text{rank}(x) = \deg(x), \quad x \in \tilde{G}, \quad \text{rank}(\lambda) = \deg(x\lambda) - \deg(x),$$

$$\text{rank}(u_i) = \alpha - \epsilon_i, \quad \text{rank}(\partial) = \alpha,$$

$$\text{rank}(\beta) = 0 \quad (\beta = \text{constant}),$$

where $\deg(R) = \alpha$ and $\deg(e_i) = \epsilon_i$.

Denote

$$g_+ = \sum_{n \geq \pi} g_n, \quad g_- = \sum_{n < \pi} g_n,$$

and call g_+ the positive part of g , where $\pi \in \mathbb{Z}$ is a properly chosen integer.

The DLW hierarchy corresponding to equation (1.1) derived from the spectral problem (1.2) is now given to be a basis of iterated algebra over $\mathbb{C}$ as follows:

$$h(n) = \begin{pmatrix} \frac{1}{2}\lambda^n & 0 \\ 0 & -\frac{1}{2}\lambda^n \end{pmatrix}, \quad e(n) = \begin{pmatrix} 0 & \lambda^n \\ 0 & 0 \end{pmatrix}, \quad f(n) = \begin{pmatrix} 0 & 0 \\ \lambda^n & 0 \end{pmatrix}, \quad (2.2)$$

such that the following equalities,

$$[h(m), e(n)] = e(m+n), \quad [f(m), h(n)] = f(m+n),$$

$$[e(m), f(n)] = 2h(m+n),$$

hold. The gradation for the basis is defined by

$$\deg h(n) = \deg e(n) = \deg f(n) = n.$$

From Eqs. (2.2), the U in (1.2) can be expressed as

$$U = -h(1) + f(0) + uh(0) - ve(0),$$

and we have

$$\deg h(1) = 1, \quad \deg e(0) = \deg f(0) = 0,$$

$$\text{rank}(\lambda) = \text{rank}(\partial) = \text{rank}(u) = \text{rank}(v) = 1.$$

The adjoint equation

$$V_x = [U, V] = UV - VU$$

with

$$V = ah(0) + be(0) - cf(0)$$

can be solved and we obtain

$$a_x = 2vc - 2b, \quad b_x = -b\lambda + ub + va,$$

$$c_x = c\lambda + uc + a. \quad (2.3)$$

Substituting the Laurent expansion $a = \sum a_m \lambda^{-m}$, $b = \sum b_m \lambda^{-m}$, $c = \sum c_m \lambda^{-m}$ into (2.3), we obtain the following recursive formulas:

$$a_{mx} = 2vc_m - 2b_m,$$

$$b_{mx} = -b_{m+1} + ub_m + va_m,$$

$$c_{mx} = c_{m+1} + uc_m + a_m. \quad (2.4)$$

Taking $a_0 = 0$, $c_0 = \alpha$, $b_0 = \alpha v$, we further derive from (2.4) that

$$\begin{pmatrix} \frac{1}{2}a_{m+1} \\ c_{m+1} \end{pmatrix} = L \begin{pmatrix} \frac{1}{2}a_m \\ c_m \end{pmatrix},$$

where L is the matrix operator defined by

$$L = \begin{pmatrix} -\partial + \partial^{-1}u\partial & \partial^{-1}v\partial + v \\ 2 & \partial + u \end{pmatrix}.$$

Set

$$(\lambda^n V)_+ = \sum_{m=0}^n [a_m h(n-m) + b_m e(n-m) - c_m f(n-m)].$$

Then, $V_x = [U, V]$ implies that

$$-(\lambda^n V)_{+x} + [U, (\lambda^n V)_+] = -(\lambda^n V)_{-x} + [U, (\lambda^n V)_-]. \quad (2.5)$$

Notice that all the terms on the left-hand side of (2.5) have degrees ≥ 0 , whereas all the terms on the right-hand side of (2.5) have degrees ≤ 0 . Therefore, we have

$$-(\lambda^n V)_{+x} + [U, (\lambda^n V)_+] \in Ch(0) + Ce(0) + Cf(0). \quad (2.6)$$

In other words, we only need to calculate those terms that contain $h(0), e(0), f(0)$. Direct calculation shows that

$$-(\lambda^n V)_{+x} + [U, (\lambda^n V)_+] = b_{n+1}e(0) + c_{n+1}f(0).$$

To cancel the term $c_{n+1}f(0)$, we introduce $\Delta_n = c_{n+1}h(0)$. It is then easy to prove that, for $V^{(n)} = (\lambda^n V)_+ + \Delta_n$, we have

$$-(V^{(n)})_{+x} + [U, (V^{(n)})_+] = -c_{n+1}h(0) - \frac{1}{2}a_{n+1}e(0).$$

Then zero curvature equation $U_t - (V^{(n)})_{+x} + [U, (V^{(n)})_+]$ gives the following DLW hierarchy:

$$u_t = c_{n+1}x, \quad v_t = \frac{1}{2}a_{n+1}x, \quad n = 1, 2, \dots \quad (2.7)$$

We note here that the first two representative systems in the hierarchy (2.7) are

$$u_t = \alpha u_x, \quad v_t = \alpha v_x,$$

$$u_t = \alpha(u_{xx} + 2v_x + 2uu_x), \quad v_t = \alpha(-v_{xx} + 2u_x v + 2uv_x).$$

The second system is exactly the DLW equation (1.1) in the case when $\alpha=1$.

The following theorem serves to develop the Hamiltonian structure of the hierarchy (2.7).

Theorem 2.1: The hierarchy (2.7) possesses the bi-Hamiltonian structure

$$w_t = J \frac{\delta H_n}{\delta w} = K \frac{\delta H_{n-1}}{\delta w}, \quad (2.8)$$

where the vector function w , the Hamiltonian operators J, K and the Hamiltonian function are given by

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad K = JL = \begin{pmatrix} 2\partial & \partial^2 + \partial u \\ -\partial^2 + u\partial & v\partial + \partial v \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad H_n = \frac{1}{2}a_{n+2}, \quad n = 0, 1, \dots \quad (2.9)$$

Proof: We introduce

$$G_{n+1} = \begin{pmatrix} \frac{1}{2}a_{n+1} \\ c_{n+1} \end{pmatrix}, \quad (2.10)$$

then hierarchy (2.7) can then be rewritten in the form of

$$w_t = JG_{n+1} = KG_n = JL^{n+1}G_0 = J(J^{-1}K)^{n+1}G_0, \quad n = 1, 2, \dots, \quad (2.11)$$

where $G_0 = (0, 1)^T$.

We take the Killing–Cartan form $\langle A, B \rangle$ to be $\text{tr}(AB)$. From direct calculation we then obtain

$$\left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = -\frac{1}{2}a, \quad \left\langle V, \frac{\partial U}{\partial u} \right\rangle = \frac{1}{2}a, \quad \left\langle V, \frac{\partial U}{\partial v} \right\rangle = c.$$

By using the result of trace identity,⁶ we have

$$\frac{\delta}{\delta w} \left(-\frac{1}{2}a \right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \left[\lambda^\gamma \left(\frac{1}{2}a, c \right)^T \right].$$

Substituting

$$a = \sum_{n=0}^{\infty} a_n \lambda^{-n}, \quad c = \sum_{n=0}^{\infty} c_n \lambda^{-n}$$

into the above equation yields

$$\frac{\delta}{\delta w} \left(-\frac{1}{2}a_{n+1} \right) = (\gamma - n)G_n, \quad (2.12)$$

which holds for all $n=0, 1, 2, \dots$. The condition $n=0$ in (2.12) gives $\gamma=1$. Therefore, we obtain

$$G_{n+1} = \frac{\delta H_n}{\delta w}, \quad (2.13)$$

where H_n is given by (2.9).

Combining (2.11) with (2.13) we then obtain the desired Hamiltonian formulation (2.8) for the DLW hierarchy (2.7).

It is crucial to prove the existence of infinite involutive conserved densities. The inner product between two functions f and g is defined by

$$(f, g) = \int fg \, dx,$$

and the Poisson bracket is defined by

$$\{f, g\} = \left(\frac{\delta f}{\delta w}, J \frac{\delta g}{\delta w} \right).$$

Finally, the integrability of the hierarchy (2.7) or (2.8) is given in the following Theorem 2.2.

Theorem 2.2: The Hamiltonian functions $\{H_n\}_{n=0}^{\infty}$ defined by (2.9) constitute common conserved densities for the whole hierarchy (2.8). In other words, it is an integrable Hamiltonian system in Liouville sense.

Proof: It is easy to see that $JL = L^*J$. We then have

$$\begin{aligned} \{H_n, H_m\} &= \left(\frac{\delta H_n}{\delta w}, J \frac{\delta H_m}{\delta w} \right) = (L^{n+1}G_0, JL^{m+1}G_0) = (L^{n+1}G_0, L^*JL^mG_0) = (L^{n+2}G_0, JL^mG_0) \\ &= \{H_{n+1}, H_{m-1}\}. \end{aligned}$$

Repeating the above argument gives

$$\{H_n, H_m\} = \{H_m, H_n\} = \{H_{m+n}, H_0\}. \quad (2.14)$$

On the other hand, we have

$$\{H_m, H_n\} = (L^{m+1}G_0, JL^{n+1}G_0) = (J^*L^{m+1}G_0, L^{n+1}G_0) = -\{H_n, H_m\}. \quad (2.15)$$

Combining (2.14) with (2.15) gives

$$\{H_m, H_n\} = 0,$$

which implies that $\{H_n\}$ are in involution. Furthermore, we have

$$\left(\int H_n dx \right)_t = \left(\frac{\delta H_n}{\delta u}, u_t \right) = \left(\frac{\delta H_n}{\delta u}, J \frac{\delta H_m}{\delta u} \right) = \{H_n, H_m\} = 0.$$

This implies that $\{H_n\}$ are also conserved densities.

III. ZERO CURVATURE AND LAX REPRESENTATIONS

The zero curvature and Lax representations are two different commutator representations of nonlinear evolution equations. They play an important role in the investigation of integrable systems. To obtain the two commutator representations for the DLW hierarchy (2.7), based on the general spectral problem (2.1) we first propose an approach to derive a hierarchy of nonlinear evolution equations from which their zero representation can be obtained. Recall that the Gateaux derivative of a spectral operator $U(u, \lambda)$ in direction $\xi = (\xi_1, \xi_2, \dots, \xi_p)^T$ at point u is defined by

$$U'(\xi) = \frac{d}{d\varepsilon} U(u + \varepsilon \xi) \Big|_{\varepsilon=0}.$$

Denote

$$\nabla \lambda = \left(\frac{\partial \lambda}{\partial u_1}, \dots, \frac{\partial \lambda}{\partial u_p} \right)^T$$

to be the functional of λ . If there exist two $p \times p$ matrix integro-differential operators K and J depending on u such that

$$K \nabla \lambda = \lambda^k J \nabla \lambda,$$

where k being a constant, then K and J are called a pair of Lenard's operators.

By using the pair of Lenard's operators K and J , we can define the Lenard's recursion sequence by

$$G_0 \in \text{Ker } J = \{G | JG\}, \quad KG_{j-1} = JG_j, \quad j = 1, 2, \dots$$

The corresponding field $X_j = JG_j$ produces a hierarchy of nonlinear evolution equations connected with the spectral problem (2.1) as follows:

$$u_t = X_n(u) = J(J^{-1}K)^n G_0, \quad n = 1, 2, \dots \quad (3.1)$$

The zero curvature representation of the hierarchy (3.1) can now be constructed using the following theorem.

Theorem 3.1: Suppose that

- (a) $G_0 \in \text{Ker } J = \{G | JG\}$.
- (b) For arbitrary given vector function $G = (G^{(1)}, \dots, G^{(p)})^T$, the following operator equation

$$V_x - [U, V] = U'(KG) - \lambda^\beta U'(JG) \quad (3.2)$$

has a solution $V = V(G)$.

Let

$$A_n = \sum_{j=1}^n V_j \lambda^{\beta(n-j)} = \sum_{j=0}^n V(G_j) \lambda^{\beta(n-j)},$$

where $V_j = V(G_j)$. The hierarchy (3.1) then possesses the following zero curvature representation:

$$U_t - A_{nx} + [U, A_n] = 0 \quad (3.3)$$

and also the following Lax pair:

$$\psi_x = U\psi, \quad \psi_t = A_n\psi.$$

Proof: Since $G_0 \in \text{Ker } J = \{G \mid JG = 0\}$, we have

$$\begin{aligned} A_{nx} - [U, A_n] &= \sum_{j=1}^n (V_{jx} - [U, V_j])\lambda^{\beta(n-j)} = \sum_{j=1}^n [U'(KG_{j-1}) - \lambda^\beta U'(JG_{j-1})]\lambda^{\beta(n-j)} \\ &= \sum_{j=1}^n U'(JG_j)\lambda^{\beta(n-j)} - \sum_{j=0}^{n-1} U'(JG_j)\lambda^{\beta(n-j+1)} = U'(JG_n) - U'(JG_0)\lambda^{\beta(n+1)} = U'(X_n), \end{aligned}$$

where the vector field $X_n = JG_n = KG_{n-1}$.

Since $U_t = U'(u_t)$, it follows that

$$U_t - A_{nx} + [U, A_n] = U'(u_t) - U'(X_n) = U'(u_t - X_n) = 0.$$

The construction of the zero representation for the hierarchy (2.7) is given in the following theorem. We first need the following lemmas.

Lemma 3.1: Let λ be an eigenvalue of equation (1.2). The functional gradient $\nabla\lambda$ of λ is then given by

$$\nabla\lambda = \left(\int \psi_1 \psi_2 \right)^{-1} (\psi_1 \psi_2, -\psi_2^2)^T,$$

where K and J are the pair of Lenard's operators

$$K \nabla\lambda = \lambda J \nabla\lambda, \quad (3.4)$$

and $\{G_j\}$ is the Lenard's sequence

$$KG_{j-1} = JG_j, \quad j = 1, 2, \dots, \quad (3.5)$$

where K and J are defined by (2.9) and (2.10).

Proof: The result can be directly computed from (1.2).

Lemma 3.2: The differential map of $U(u, \lambda)$ in (1.2) is given by

$$U'(\xi) = \begin{pmatrix} \frac{1}{2}\xi_1 & -\xi_2 \\ 0 & -\frac{1}{2}\xi_1 \end{pmatrix}, \quad \xi = (\xi_1, \xi_2)^T,$$

so that $U'(\xi)$ is an injective homomorphism.

Lemma 3.3: If $\beta=1$ in (3.2), then for arbitrary given vector function $G=(G^{(1)}, G^{(2)})^T$, the following operator equation

$$V_x - [U, V] = U'(KG) - \lambda U'(JG) \quad (3.6)$$

has a solution

$$V = V(G) = \begin{pmatrix} -\frac{1}{2}\lambda G^{(2)} + \frac{1}{2}(G_x^{(2)} + uG^{(2)}) & G_x^{(1)} - vG^{(2)} \\ G^{(2)} & \frac{1}{2}\lambda G^{(2)} - \frac{1}{2}(G_x^{(2)} + uG^{(2)}) \end{pmatrix}. \quad (3.7)$$

Proof: By representing the commutator $[U, V]$ as $U=U_0+U_1\lambda$ and $V=V_0+V_1\lambda$ such that

$$U_0 = \begin{pmatrix} \frac{1}{2}u & -v \\ 1 & -\frac{1}{2}u \end{pmatrix}, \quad U_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

$$V_0 = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad V_1 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix},$$

where A, B, C, D are four undetermined functions, we have

$$\begin{aligned} [U, V] &= [U_0, V_0] + (U_0, V_1] + [U_1, V_0])\lambda + [U_1, V_1]\lambda^2 = [U_0, V_0] + (U_0, V_1] + [U_1, V_0])\lambda \\ &= \begin{pmatrix} -vC - B & 2vA + uB \\ 2A - uC & B + vC \end{pmatrix} + \begin{pmatrix} 0 & 2vD - B \\ 2D + C & 0 \end{pmatrix} \lambda, \end{aligned} \quad (3.8)$$

and

$$KG = \begin{pmatrix} 2G_x^{(1)} + G_{xx}^{(2)} + uG_x^{(2)} + u_x G^{(2)} \\ -G_{xx}^{(1)} + uG_x^{(1)} + 2vG_x^{(2)} + v_x G^{(2)} \end{pmatrix}, \quad JG = \begin{pmatrix} G_x^{(2)} \\ G_x^{(1)} \end{pmatrix}. \quad (3.9)$$

Substituting (3.8) and (3.9) into (3.6) and comparing the corresponding elements in both sides of the matrices, we can determine the values of A, B, C , and D from which the solution V as given in (3.7) can be obtained.

Finally, from Theorem 3.1 we have the following.

Theorem 3.2: Let

$$A_n = \sum_{j=1}^n V_j \lambda^{n-j} = \sum_{j=0}^n V(G_j) \lambda^{n-j},$$

where $V_j = V(G_j)$ is given by (3.7). The DLW hierarchy (2.7) possesses the following zero curvature representation:

$$U_t - A_{nx} + [U, A_n] = 0, \quad (3.10)$$

and the Lax pair

$$\psi_x = U\psi, \quad \psi_t = A_n\psi.$$

Similarly we can obtain the Lax representation of the hierarchy (2.7) through the following lemmas and theorem.

Lemma 3.4: The spectral problem (1.2) is equivalent to

$$L\psi = \begin{pmatrix} -2\partial + u & -2v \\ -2 & 2\partial + u \end{pmatrix} \psi = \lambda\psi, \quad (3.11)$$

where the differential map of $L = L(u, v, \lambda)$ is

$$L'(\xi) = \begin{pmatrix} \xi_1 & -2\xi_2 \\ 0 & \xi_1 \end{pmatrix},$$

and $L'(\xi)$ is an injective homomorphism.

Proof: Direct calculation from (1.2).

Lemma 3.5: For arbitrary given vector function $G = (G^{(1)}, G^{(2)})^T$, the following operator equation

$$[V, L] = L'(KG) - L'(JG)L \quad (3.12)$$

has a solution

$$V = V(G) = \begin{pmatrix} \frac{1}{2}(G_x^{(2)} + uG^{(2)}) & G_x^{(1)} - vG^{(2)} \\ G^{(2)} & -\frac{1}{2}(G_x^{(2)} + uG^{(2)}) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}G^{(2)} & 0 \\ 0 & -\frac{1}{2}G^{(2)} \end{pmatrix} L. \quad (3.13)$$

Proof: Representing the commutator $[V, L]$ of $L = L_0 + L_1\partial_x$ and $V = V_0 + V_1\partial_x$ such that

$$L_0 = \begin{pmatrix} u & -2v \\ -2 & u \end{pmatrix}, \quad U_1 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

$$V_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad U_1 = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},$$

where A, B, C, D are four undetermined functions, we have

$$\begin{aligned} [V, L] &= [V_0, L_0] + [V_0, L_1 \partial_x] + [V_1 \partial_x, L_0] + [V_1 \partial_x, L_1 \partial_x] \\ &= [V_0, L_0] + V_1 L_{0x} - L_1 V_{0x} + ([V_0, L_1] + [V_1, L_0] + V_1 L_{1x} - L_1 V_{1x}) \partial_x \\ &= \begin{pmatrix} -2B + 2vC + u_x E + 2A_x & -2v(A - D) - 2v_x E + 2B_x \\ 2(A - D) - 2C_x & 2B - 2vC + u_x F - 2D_x \end{pmatrix} \\ &\quad + \begin{pmatrix} 2E_x & -2v(E - F) + 4B \\ 2(E - F) - 4C & -2F_x \end{pmatrix} \partial_x. \end{aligned} \quad (3.14)$$

We also have

$$\begin{aligned} L'[KG] - L'(JG)L &= \begin{pmatrix} 2G_x^{(2)} & 4G_x^{(1)} \\ 0 & -2G_x^{(2)} \end{pmatrix} \partial_x \\ &\quad + \begin{pmatrix} 2G_x^{(1)} + G_{xx}^{(2)} + u_x G^{(2)} - 4G_x^{(1)} & 2G_{xx}^{(1)} - 2vG_x^{(2)} - 2v_x G^{(2)} \\ 2G_x^{(2)} & 2G_x^{(1)} + G_{xx}^{(2)} + u_x G^{(2)} \end{pmatrix}. \end{aligned} \quad (3.15)$$

Substituting (3.14) and (3.15) into (3.12) and comparing the corresponding elements on both sides of the matrices, we then obtain

$$A = -D = \frac{1}{2}G_x^{(2)}, \quad B = G_x^{(1)}, \quad C = 0, \quad E = F = G^{(2)}.$$

Since $\partial_x = L_1^{-1}(L - L_0)$, we have

$$\begin{aligned} V &= V(G) = \begin{pmatrix} \frac{1}{2}G_x^{(2)} & G_x^{(1)} \\ 0 & -\frac{1}{2}G_x^{(2)} \end{pmatrix} + \begin{pmatrix} (2) & 0 \\ 0 & G^{(2)} \end{pmatrix} \partial_x \\ &= \begin{pmatrix} \frac{1}{2}(G_x^{(2)} + uG^{(2)})G_x^{(1)} - vG^{(2)} & \\ G^{(2)} & -\frac{1}{2}(G_x^{(2)} + uG^{(2)}) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}G^{(2)} & 0 \\ 0 & \frac{1}{2}G^{(2)} \end{pmatrix} L. \end{aligned}$$

Theorem 3.3: Let

$$B_n = \sum_{j=1}^n V_j L^{n-j} = \sum_{j=0}^n V(G_j) L^{n-j}, \quad (3.16)$$

where $V_j = V(G_j)$, $G_j = (G_j^{(1)}, G_j^{(2)})^T$, and the matrix function V satisfies (3.13). The DLW hierarchy (2.7) possesses the following Lax representation:

$$L_t = [B_n, L], \quad (3.17)$$

and the Lax pair

$$L\psi = \lambda\psi, \quad \psi_t = B_n\psi.$$

Proof: From (3.12) and (3.16), we have

$$\begin{aligned}
[B_m, L] &= \sum_{j=1}^n [V_j, L] L^{n-j} = \sum_{j=1}^n [L'(KG_{j-1}) - L'(JG_{j-1})L] L^{n-j} \\
&= \sum_{j=1}^n L'(JG_j) L^{n-j} - \sum_{j=0}^{n-1} L'(JG_j) L^{n-j} = L'(JG_n) - L'(JG_0) = L'(X_n).
\end{aligned}$$

Since $L_t = L'(u_t)$, it follows that

$$L_t - [B_m, L] = L'(u_t) - L'(X_m) = L'(u_t - X_m) = 0.$$

The proof is completed.

IV. A FINITE DIMENSIONAL HAMILTONIAN SYSTEM

To construct the algebro-geometric solution of DLW hierarchy, in this section we investigate the finite dimensional Hamiltonian systems associated with the spectral problem (1.2) through a nonlinearization approach.^{12,13} Let $\lambda_j, j=1, \dots, N$ be N different eigenvalues of equation (1.2), and (p_j, q_j) be the associated eigenfunctions, i.e., we consider the following N eigenvalues:

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}(\lambda_j - u) & -v \\ 1 & \frac{1}{2}(\lambda_j - u) \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix}. \quad (4.1)$$

Denote $p = (p_1, p_2, \dots, p_N)^T$, $q = (q_1, q_2, \dots, q_N)^T$, and $\wedge = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. From the Bargmann constraint

$$G_1 = \begin{pmatrix} v \\ u \end{pmatrix} = \sum_{j=1}^N \nabla \lambda_j = \begin{pmatrix} \langle p, q \rangle \\ -\langle q, q \rangle \end{pmatrix}, \quad (4.2)$$

we then obtain

$$w = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\langle q, q \rangle \\ \langle p, q \rangle \end{pmatrix} \equiv h(p, q). \quad (4.3)$$

From the constraint (4.3), the equation (4.1) can further be nonlinearized into the following finite-dimensional Hamiltonian system:

$$\begin{aligned}
p_x &= -\frac{1}{2} \wedge p - \frac{1}{2} \langle q, q \rangle p - \langle p, q \rangle q = -\frac{\partial H}{\partial q}, \\
q_x &= p + \frac{1}{2} \wedge q + \frac{1}{2} \langle q, q \rangle q = \frac{\partial H}{\partial p},
\end{aligned} \quad (4.4)$$

whose Hamiltonian function H is given by

$$H = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \langle \wedge p, q \rangle + \frac{1}{2} \langle q, q \rangle \langle p, q \rangle.$$

In the following, we proceed to show that the Hamilton system (4.4) is completely integrable in the Liouville sense. The Poisson bracket of two functions in symplectic space $(\mathbb{R}^{2N}, dp \wedge dq)$ is defined as

$$(F, G) = \sum_{j=1}^N \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right) = \left\langle \frac{\partial F}{\partial q}, \frac{\partial G}{\partial p} \right\rangle - \left\langle \frac{\partial F}{\partial p}, \frac{\partial G}{\partial q} \right\rangle,$$

which is skew-symmetric, bilinear, and satisfies the Jacobi identity. In particular, F and G are called involution if $(F, G) = 0$.

Consider a bilinear function $Q_\lambda(p, q) = \langle (\lambda I - \wedge)^{-1} p, q \rangle$ such that

$$G_\lambda = G_0 + \sum_{j=1}^N \frac{\nabla \lambda_j}{\lambda - \lambda_j} = \begin{pmatrix} Q_\lambda(p, q) \\ 1 - Q_\lambda(q, q) \end{pmatrix}. \quad (4.5)$$

From (3.4) and (4.5), we have

$$V_\lambda = V(G_\lambda) = \begin{pmatrix} -\frac{1}{2}\lambda - Q_\lambda(p, q) & -\langle p, q \rangle + Q_\lambda(p, p) \\ 1 - Q_\lambda(q, q) & \frac{1}{2}\lambda + Q_\lambda(p, q) \end{pmatrix}.$$

Let $F_\lambda = \det V_\lambda$. Then, we have

$$\begin{aligned} F_\lambda &= -\frac{1}{4}\lambda^2 - Q_\lambda(\wedge p, q) + Q_\lambda(p, p) + \langle p, p \rangle Q_\lambda(q, q) + \begin{vmatrix} Q_\lambda(p, p) & Q_\lambda(p, q) \\ Q_\lambda(p, q) & Q_\lambda(q, q) \end{vmatrix} \\ &= -\frac{1}{4}\lambda^2 + \sum_{m=0}^{\infty} F_m \lambda^{-m-1}, \end{aligned} \quad (4.6)$$

where

$$F_0 = 2H,$$

$$F_m = -\langle \wedge^{m+1} p, p \rangle + \langle p, q \rangle \langle \wedge^m q, q \rangle + \sum_{m=0}^{m-1} \begin{vmatrix} \langle \wedge^j p, p \rangle & \langle \wedge^{m-1-j} p, q \rangle \\ \langle \wedge^j p, q \rangle & \langle \wedge^{m-1-j} q, q \rangle \end{vmatrix}.$$

By considering that F_λ is a Hamiltonian in the symplectic space $(\mathbb{R}^{2N}, dp \wedge dq)$, the canonical equations of the F_λ -flow can be computed as

$$\frac{d}{d\tau_\lambda} \begin{pmatrix} p_k \\ q_k \end{pmatrix} = I \nabla F_\lambda = \begin{pmatrix} -\partial F_\lambda / \partial q_k \\ \partial F_\lambda / \partial p_k \end{pmatrix} = W(\lambda, \lambda_k) \begin{pmatrix} p_k \\ q_k \end{pmatrix},$$

where

$$W(\lambda, \lambda_k) = \frac{2}{\lambda - \lambda_k} V_\lambda + (1 - Q_\lambda(p, q)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 4.1: The Lax matrix V_μ satisfies the Lax equation (4.7) along the t_λ -flow:

$$\frac{dV_\mu}{d\tau_\lambda} = [W(\lambda, \mu), V_\mu] \quad (4.7)$$

and

$$(F_\mu, F_\lambda) = 0, \quad \forall \lambda, \mu \in \mathbb{C}, \quad (4.8)$$

$$(F_j, F_k) = 0, \quad \forall j, k = 0, 1, \dots. \quad (4.9)$$

Proof: Notice that

$$Q_\lambda(\wedge \xi, \eta) = \lambda Q_\lambda(\wedge \xi, \eta) - \langle \xi, \eta \rangle,$$

$$\langle (\mu I - \wedge)^{-1} (\lambda I - \wedge)^{-1} \xi, \eta \rangle = \frac{1}{\mu - \lambda} (Q_\lambda(\xi, \eta) - Q_\mu(\xi, \eta)),$$

a direct calculation shows that (4.7) holds. From the Lax equation (4.7) we have

$$\begin{aligned}
(F_\mu, F_\lambda) &= \sum_{j=1}^N \left(\frac{\partial F_\mu}{\partial q_j} \frac{\partial F_\lambda}{\partial p_j} - \frac{\partial F_\mu}{\partial p_j} \frac{\partial F_\lambda}{\partial q_j} \right) = \sum_{j=1}^N \left(\frac{\partial F_\mu}{\partial q_j} \frac{dq_j}{dt_\lambda} + \frac{\partial F_\mu}{\partial p_j} \frac{dp_j}{dt_\lambda} \right) \\
&= \frac{dF_\mu}{d\tau_\lambda} = \frac{d}{d\tau_\lambda} (\det V_\mu) = \frac{d}{d\tau_\lambda} \left(-\frac{1}{2} \text{tr} V_\mu^2 \right) = -\text{tr} \left(V_\mu \frac{dV_\mu}{d\tau_\lambda} \right) \\
&= -\text{tr} V_\mu \text{tr} [W(\lambda, \mu), V_\mu] = 0.
\end{aligned}$$

Substituting (4.6) into (4.8) and using the coefficients of λ and μ , we then obtain (4.9).

We conclude the above results in the following theorem.

Theorem 4.1: The finite dimensional Hamiltonian system defined by (4.4) is completely integrable in Liouville sense in the symplectic space $(\mathbb{R}^{2N}, dp \wedge dq)$.

Theorem 4.2: Let (p, q) be a solution of the Hamiltonian system (4.4). Then, u and v defined by (4.3) satisfy the following stationary DLW equation

$$X_N + c_1 X_{N-2} + \cdots + c_{N-1} X_0 = 0 \quad (4.10)$$

with suitably chosen constants $c_1, \dots, c_{N-1}$.

Proof: Operating with $(J^{-1}K)^{k-1}$ upon the expression of (4.2), we get

$$G_k + \beta_1 G_{k-2} + \cdots + \beta_{k-1} G_0 + \beta_k G_{-1} = \sum_{j=1}^N \lambda_j^k \nabla \lambda_j, \quad (4.11)$$

where $\beta_1, \dots, \beta_k$ are arbitrary constants, $G_{-1} = (0, 0)^T$, and $J^{-1}KG_{-1} = G_0$.

Consider the polynomial

$$p(\lambda) = \prod_{j=1}^N (\lambda - \lambda_j) = \lambda^N + p_1 \lambda^{N-1} + \cdots + p_N.$$

Applying the operator $J \sum_{k=1}^N p_{N-k}$ on (4.11), we obtain (4.10), where $c_1, \dots, c_N$ depend on $\beta_1, \dots, \beta_k$ and $\lambda_1, \dots, \lambda_N$.

Introduce the generating function $\{g_k\}$,

$$g_\lambda = G_0 + \sum_{k=0}^{\infty} G_k \lambda^{-k}, \quad (4.12)$$

which satisfies

$$(K - \lambda J)g_\lambda = 0,$$

we have

$$V_x(g_\lambda) - [U, V(g_\lambda)] = U'((K - \lambda J)g_\lambda) = 0.$$

This implies that

$$\frac{\partial}{\partial x} (\det V(g_\lambda)) = 0.$$

Hence, we have

$$\det V(g_\lambda) = -\frac{1}{4} \lambda^2. \quad (4.13)$$

Define a new set of integrals $\{H_k\}$, respectively, by

$$H_0 = \frac{1}{2} F_0, \quad H_1 = -\frac{1}{2} F_1, \quad H_2 = -\frac{1}{2} F_2,$$

$$H_m = -\frac{1}{2}F_m + 2 \sum_{j=0}^{m-3} H_j H_{m-j-3}, \quad m \geq 3.$$

Equation (4.6) can then be transformed into the following equivalent form:

$$-\frac{4}{\lambda^2}F_\lambda = (1 - 4H_\lambda)^2, \quad (4.14)$$

where

$$H_\lambda = \sum_{m=1}^{\infty} H_{m-1} \lambda^{-m-1}.$$

The involutivity of $\{H_k\}$ is based on the equality

$$\{H_\mu, H_\lambda\} = \frac{1}{16\sqrt{F_\lambda F_\mu}} \{F_\mu, F_\lambda\} = 0.$$

By using (4.5) and (4.11), we have

$$\begin{aligned} G_\lambda &= G_0 + \sum_{k=0}^{\infty} \lambda^{-k-1} \sum_{j=1}^N \lambda_j^k \nabla \lambda_j, \\ &= G_0 + \sum_{k=0}^{\infty} c^{-k-1} (G_k + c_1 G_{k-2} + \cdots + c_{k-1} G_0 + c_k G_{-1}), \\ &= c_\lambda g_\lambda + \sum_{k=0}^{\infty} \lambda^{-k-1} c_k G_{-1}, \end{aligned}$$

where

$$c_\lambda = 1 + \sum_{k=0}^{\infty} c_{k+2} \lambda^{-k-2}. \quad (4.15)$$

Using (4.6) and (4.14) again, we have

$$V_\lambda = V \left(c_\lambda g_\lambda + \sum_{k=0}^{\infty} \lambda^{-k-1} c_k G_{-1} \right) = V(c_\lambda g_\lambda).$$

Therefore, we have

$$F_\lambda = \det V_\lambda = -\frac{1}{4} c_\lambda^2 g_\lambda^2. \quad (4.16)$$

From (4.14) and (4.16), we obtain

$$c_\lambda = 1 - 4H_\lambda.$$

Denote the variables of H_λ flow and H_k flow by t_λ and t_k , respectively. Applying the Leibniz rule of the Poisson bracket, we obtain

$$\frac{1}{2\lambda^2} \{\psi, F_\lambda\} = (1 - 4H_\lambda) \{\psi, H_\lambda\},$$

for any smooth function ψ . Thus,

$$\frac{d}{dt_\lambda} = \frac{1}{2\lambda^2(1-4H_\lambda)} \frac{d}{d\tau_\lambda} = \frac{1}{2\lambda^2 c_\lambda} \frac{d}{d\tau_\lambda}. \quad (4.17)$$

For $w=(u,v)^T=h(p,q)$, we have

$$\frac{dw}{d\tau_\lambda} = \begin{pmatrix} -2 \left\langle q, \frac{dq}{d\tau_\lambda} \right\rangle \\ \left\langle p, \frac{dq}{d\tau_\lambda} \right\rangle + \left\langle \frac{dp}{d\tau_\lambda}, q \right\rangle \end{pmatrix} = 2JG_\lambda,$$

$$\begin{aligned} \frac{dw}{dt_\lambda} &= \frac{1}{2\lambda^2 c_\lambda} \frac{dw}{d\tau_\lambda} = \frac{1}{\lambda^2 c_\lambda} JG_\lambda = \frac{1}{\lambda^2} Jg_\lambda \\ &= \sum_{k=0}^{\infty} JG_K \lambda^{-k-3} = \sum_{k=0}^{\infty} JG_K \lambda^{-k-3}. \end{aligned}$$

Finally, we state the above result in the following theorem.

Theorem 4.3: Let $(p(x, t_k), q(x, t_k))$ be a compatible solution of the h_0 and H_k flow. Then, $w(x, t_k)=h(p, q)$ solves the k th DLW equation

$$w_t = X_k(w).$$

V. ALGEBRO-GEOMETRIC SOLUTIONS

Based on results presented in Secs. III and IV, we give the construction of the algebro-geometric solutions for the hierarchy (2.7) in the following lemmas and theorem.

Lemma 5.1: The solutions u and v for the DLW hierarchy (2.7) can be expressed in the form

$$u = \sum_{k=1}^N (\lambda_k - \nu_k), \quad \partial_x \ln v = \sum_{k=1}^N (\mu_k - \nu_k), \quad (5.1)$$

where μ_k, ν_k are called the elliptic coordinates of the finite-dimensional Hamiltonian system (4.3).

Proof: We express F_λ in the form

$$F_\lambda = -V_\lambda^{11^2} - V_\lambda^{12} V_\lambda^{21} = -\frac{1}{4} \lambda^2 + \sum_{k=1}^N \frac{E_k}{\lambda - \lambda_k},$$

where

$$E_k = -\lambda_k p_k q_k + p_k^2 + \langle p, q \rangle q_k^2 + \sum_{j=1, j \neq k}^N \frac{(p_j q_k - p_k q_j)^2}{\lambda_j - \lambda_k}.$$

Let

$$F_\lambda = -\frac{b(\lambda)}{4a(\lambda)} = -\frac{R(\lambda)}{4a^2(\lambda)}, \quad (5.2)$$

$$V_{\lambda}^{12} = -\langle p, q \rangle + Q_{\lambda}(p, p) = -\langle p, q \rangle \frac{m(\lambda)}{a(\lambda)},$$

$$V_{\lambda}^{21} = 1 - Q_{\lambda}(q, q) = \frac{n(\lambda)}{a(\lambda)}, \quad (5.3)$$

where

$$a(\lambda) = \prod_{k=1}^N (\lambda - \lambda_k), \quad b(\lambda) = \prod_{k=1}^{N+2} (\lambda - \lambda_{N+k}),$$

$$m(\lambda) = \prod_{k=1}^N (\lambda - \mu_k), \quad n(\lambda) = \prod_{k=1}^N (\lambda - \nu_k),$$

$$R(\lambda) = a(\lambda)b(\lambda) = \prod_{k=1}^{2N+2} (\lambda - \lambda_k).$$

Comparing the coefficients of λ^{N-1} in (5.2) and (5.3) gives

$$\langle q, q \rangle = -\sum_{k=1}^N (\lambda_k - \nu_k), \quad \frac{\langle p, p \rangle}{\langle p, q \rangle} = \sum_{k=1}^N (\mu_k - \lambda_k). \quad (5.4)$$

By using (4.2) and (5.4), we then obtain (5.1).

From the Lax equation (4.7), we have

$$\frac{dV_{\mu}^{12}}{dt_{\lambda}} = 2(W^{11}V^{12} - W^{12}V^{11}),$$

$$\frac{dV_{\mu}^{21}}{dt_{\lambda}} = 2(W^{21}V^{11} - W^{11}V^{21}). \quad (5.5)$$

Taking $\lambda = \mu_k, \nu_k$ in (5.2) and (5.3), we get

$$V_{\mu_k}^{11} = \frac{\sqrt{R(\mu_k)}}{2a(\mu_k)}, \quad V_{\nu_k}^{11} = \frac{\sqrt{R(\nu_k)}}{2a(\nu_k)}.$$

Hence, we have

$$\frac{1}{2\sqrt{R(\mu_k)}} \frac{d\mu_k}{d\tau_{\lambda}} = \frac{m(\lambda)}{a(\lambda)(\lambda - \mu_k)m'(\mu_k)},$$

$$\frac{1}{2\sqrt{R(\nu_k)}} \frac{d\nu_k}{d\tau_{\lambda}} = -\frac{n(\lambda)}{a(\lambda)(\lambda - \nu_k)n'(\nu_k)}.$$

By using the polynomials interpolation formula, we have

$$\sum_{k=1}^N \frac{\mu_k^{N-j}}{2\sqrt{R(\mu_k)}} \frac{d\mu_k}{d\tau_{\lambda}} = \frac{\lambda^{N-j}}{a(\lambda)}, \quad \sum_{k=1}^N \frac{\mu_k^{N-j}}{2\sqrt{R(\nu_k)}} \frac{d\nu_k}{d\tau_{\lambda}} = -\frac{\lambda^{N-j}}{a(\lambda)}. \quad (5.6)$$

Define the hyperelliptic curve by

$$\Gamma: \xi^2 - 4R(\lambda) = 0, \quad (5.7)$$

with genus $g=N$ and the usual holomorphic differentials by

$$\tilde{\omega}_j = \frac{\lambda^{N-j} d\lambda}{2\sqrt{R(\lambda)}}, \quad j = 1, 2, \dots, N. \quad (5.8)$$

Denote $P(\mu_k) = (\lambda, \xi = 2\sqrt{R(\lambda)}) \in \Gamma$. Let $P_0 \in \Gamma$ be fixed. Define the quasi-Abel–Jacobi coordinates by

$$\tilde{\phi}_j = \sum_{k=1}^N \int_{P_0}^{P(\mu_k)} \tilde{\omega}_j, \quad \tilde{\psi}_j = \sum_{k=1}^N \int_{P_0}^{P(\nu_k)} \tilde{\omega}_j, \quad j = 1, 2, \dots, N. \quad (5.9)$$

Then, (5.6) can be represented in the form

$$\frac{d\tilde{\phi}_j}{d\tau_\lambda} = \frac{\lambda^{N-j}}{a(\lambda)}, \quad \frac{d\tilde{\psi}_j}{d\tau_\lambda} = -\frac{\lambda^{N-j}}{a(\lambda)}.$$

Let $a_1, b_1, \dots, a_N, b_N$ be the canonical basis of cycles on the Γ and

$$C = (A_{jk})_{N \times N}^{-1}, \quad A_{jk} = \int_{a_k} \tilde{\omega}_j.$$

Define the normalized holomorphic differential by

$$\omega_s = \sum_{j=1}^N C_{sj} \tilde{\omega}_j, \quad \omega = (\omega_1, \dots, \omega_N)^T = C \tilde{\omega}.$$

Then, we have

$$\int_{a_k} \omega_s = \delta_{sk}, \quad \int_{b_k} \omega_s = B_{sk},$$

where the matrix $B = (B_{sk})$ is symmetric with positive-definite imaginary part. This is used to define the Riemann theta function of Γ as

$$\theta(\zeta) = \sum_{z \in \mathbb{Z}^N} \exp(\pi i \langle Bz, z \rangle + 2\pi i \langle \zeta, z \rangle), \quad \zeta \in C^N.$$

The Abel map $A(P)$ and the Abel–Jacobi coordinate are defined as

$$A(P) = \int_{P_0}^P \omega, \quad A\left(\sum n_k P_k\right) = \sum n_k A(P_k),$$

$$\phi = A\left(\sum_{k=1}^N P(\mu_k)\right) = \sum_{k=1}^N \int_{P_0}^{P(\mu_k)} \omega = C \tilde{\phi},$$

$$\psi = A\left(\sum_{k=1}^N P(\nu_k)\right) = \sum_{k=1}^N \int_{P_0}^{P(\nu_k)} \omega = C \tilde{\psi}.$$

Let $S_k = \lambda_1^k + \dots + \lambda_{2N+2}^k$, and $\tilde{R}(\lambda^{-1}) = \prod_{j=1}^{2N+2} (1 = \lambda_j \lambda^{-1})$. The coefficients in

$$\frac{1}{\sqrt{\tilde{R}(\lambda^{-1})}} = \sum_{k=1}^{\infty} \Lambda_k \lambda^{-k}$$

are then given by

$$\Lambda_0 = 1, \quad \Lambda_1 = \frac{1}{2}S_1, \quad \Lambda_k = \frac{1}{2k} \left(S_k + \sum_{i+j=k, i, j \geq 1} S_i \Lambda_j \right).$$

From (5.2), we obtain

$$\sqrt{R(\lambda)} = \lambda a(\lambda) c_\lambda,$$

and the lemma is proved.

Lemma 5.2: Let C_k be the k th column vector of the matrix C . Under the Abel–Jacobi coordinate system, we have straightened out the flows to

$$\frac{d\phi}{dt_\lambda} = \sum_{k=1}^{\infty} \Omega_{k-1} \lambda^{-k-1}, \quad \frac{d\psi}{dt_\lambda} = - \sum_{k=1}^{\infty} \Omega_{k-1} \lambda^{-k-1}, \quad (5.10)$$

$$\frac{d\phi}{dt_k} = \Omega_{k-1}, \quad \frac{d\psi}{dt_k} = - \Omega_{k-1}, \quad (5.11)$$

where

$$\Omega_0 = \frac{1}{2} \Lambda_0 C_1, \quad \Omega_1 = \frac{1}{2} (\Lambda_1 C_1 + \Lambda_0 C_2),$$

$$\Omega_k = \frac{1}{2} (\Lambda_k C_1 + \cdots + \Lambda_0 C_{k+1}), \quad k \leq N-1,$$

$$\Omega_k = \frac{1}{2} (\Lambda_k C_1 + \cdots + \Lambda_{k-N+1} C_N), \quad k \geq N-1.$$

Proof: Notice that

$$\begin{aligned} \frac{d\phi}{dt_\lambda} &= \frac{1}{2\lambda^2 c_\lambda} \frac{d\phi}{d\tau_\lambda} = \frac{1}{2\lambda^2 c_\lambda} C \frac{d\tilde{\phi}}{d\tau_\lambda} \\ &= \frac{\lambda^{N-1}}{2\sqrt{R(\lambda)}} (C_1 \lambda^{-1} + \cdots + \cdots C_N \lambda^{-N}) \\ &= \frac{1}{2\lambda^2 \sqrt{\tilde{R}(\lambda^{-1})}} (C_1 \lambda^{-1} + \cdots + \cdots C_N \lambda^{-N}) \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \Lambda_k \lambda^{-k-2} \sum_{j=1}^N C_j \lambda^{-j} = \sum_{k=0}^{\infty} \Omega_k \lambda^{-k-3}. \end{aligned}$$

Similarly, we can obtain the second formula of (5.10) and compare the coefficients of λ^{-k-1} in (5.10) to obtain (5.11).

From (5.11), we get

$$\phi = \phi_0 + \sum_{k=1}^{\infty} \Omega_{k-1} t_k, \quad \psi = \psi_0 - \sum_{k=1}^{\infty} \Omega_{k-1} t_k.$$

Hence, the evolution picture of the 1+1 flow is given as

$$\phi = \phi_0 + \Omega_0 x + \Omega_1 t, \quad \psi = \psi_0 - \Omega_0 x - \Omega_1 t. \quad (5.12)$$

Since $\deg R = 2N + 2$ on Γ , there are two infinite points ∞_1 and ∞_2 which are not branch points of Γ . From Riemann theorem, there exists two constants $M_1, M_2 \in C^N$ such that $\theta(A(P) - \phi - M_1)$, $\theta(A(P) - \psi - M_2)$ have exactly n zeroes at $\mu_1, \dots, \mu_N$ and $\nu_1, \dots, \nu_N$, respectively. That is,

$$\sum_{j=1}^N \mu_j = I(\Gamma) - \sum_{j=1}^2 \operatorname{Res}_{\lambda=\infty_j} \lambda d \ln \theta(A(P) - \phi - M_1), \quad (5.13)$$

$$\sum_{j=1}^N \nu_j = I(\Gamma) - \sum_{j=1}^2 \operatorname{Res}_{\lambda=\infty_j} \lambda d \ln \theta(A(P) - \psi - M_2), \quad (5.14)$$

with the constant

$$I(\Gamma) = \sum_{j=1}^N \int_{a_j} \lambda \omega_j.$$

For the same λ , there are two points on the different sheets of the Riemann surface Γ ,

$$P_+(\lambda) = (\lambda, \sqrt{R(\lambda)}), \quad P_-(\lambda) = (\lambda, -\sqrt{R(\lambda)}).$$

Under the local coordinate $z = \lambda^{-1}$ at infinity, the hyperelliptic curve $\Gamma: \xi^2 - 4R(\lambda) = 0$ in the neighborhood of infinity is expressed as $\tilde{\xi}^2 - 4\tilde{R}(z) = 0$ with $\tilde{\xi} = z^{N+1}\xi$ and $(z, 2(-1)^{s-1}\sqrt{\tilde{R}(z)})|_{z=0} = (0, 2(-1)^{s-1})$, $s = 1, 2$. Thus, we have

$$\begin{aligned} \omega = C\tilde{\omega} &= (C_1\lambda^{-1} + \dots + C_N\lambda^{-N}) \frac{\lambda^{N-1}d\lambda}{2\sqrt{R(\lambda)}} \\ &= (C_1\lambda^{-1} + \dots + C_N\lambda^{-N}) \frac{d\lambda}{2\lambda^2\sqrt{\tilde{R}(\lambda^{-1})}} \\ &= \frac{1}{2}(-1)^{s-1} \sum_{j=0}^{\infty} \Omega_k z^k dz \end{aligned}$$

and

$$A(P(z^{-1})) = \int_{P_0}^P \omega = - \int_{\infty_s}^{P_0} \omega + \int_{\infty_s}^P \omega = -\pi_s + \frac{1}{2}(-1)^{s-1} \sum_{k=0}^{\infty} \frac{1}{k+1} \Omega_k z^{k+1}.$$

Since the theta function is an even function, we have

$$\theta(A(P(z^{-1})) - \phi - M_1) = \theta(\phi + M_1 + \eta_s) + \frac{1}{2}z(-1)^{s-1} \frac{\partial}{\partial x} \theta(\phi + M_1 + \eta_s) + o(z^2). \quad (5.15)$$

From (5.13) and (5.15), we obtain

$$\begin{aligned} \sum_{j=1}^N \mu_j &= I(\Gamma) - \frac{1}{2}(-1)^{s-1} \frac{\partial}{\partial x} \theta(\phi + M_1 + \eta_s) \\ &= I(\Gamma) + \frac{1}{2} \partial_x \ln \frac{\theta(\phi + M_1 + \eta_2)}{\theta(\phi + M_1 + \eta_1)}. \end{aligned} \quad (5.16)$$

Similarly, we have

$$\sum_{j=1}^N \mu_j = I(\Gamma) + \frac{1}{2} \partial_x \ln \frac{\theta(\psi + M_2 + \pi_1)}{\theta(\psi + M_2 + \pi_2)}. \quad (5.17)$$

Substituting (5.16) and (5.17) into (5.1), we then obtain the following algebro-geometric solution for the DLW equation

$$u = \sum_{k=1}^N \lambda_k - I(\Gamma) - \frac{1}{2} \partial_x \ln \frac{\theta(\Omega_0 x + \Omega_1 t + \alpha_1)}{\theta(\Omega_0 x + \Omega_1 t + \alpha_2)},$$

$$v^2 = \frac{\theta(\Omega_0 x + \Omega_1 t + \alpha_2) \theta(\Omega_0 x + \Omega_1 t + \beta_2)}{\theta(\Omega_0 x + \Omega_1 t + \alpha_1) \theta(\Omega_0 x + \Omega_1 t + \beta_1)} v^2(0, t),$$

where

$$\alpha_l = \phi_0 + M_1 + \pi_l, \quad \beta_l = -\psi_0 - M_2 - \pi_l, \quad l = 1, 2.$$

VI. CONCLUSION

Spectral problem usually provides many important integrable properties related to nonlinear equations such as Lax pair, r -matrix, Hamiltonian structure, Darboux transformation, and algebro-geometric solution. In this paper we investigate a spectral problem associated with the DLW equation and derive a hierarchy of nonlinear evolution equations related to the DLW equation by using an iterated algebraic method. The Hamiltonian structures for the DLW hierarchy are established by using trace identity. These two methods can be extended to other kinds of hierarchies. In constructing the commutator representation, the parameter λ is found to be easier to be computed than the operator L . We thus provide a transformation to change the operator L to λ in order to obtain the Lax representation for the DLW hierarchy. Through nonlinearization technique, a finite dimensional Hamiltonian system is obtained in an explicit form. The Abel–Jacobi coordinates are introduced from which the algebro-geometric solutions for the DLW equation can be derived by resorting to the Riemann theta function. The results given in this paper can be applied to many interesting problems such as higher order constrained flow and r -matrix structure.

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